An Optimization of Phi-function for Convex Polygons

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Abstract

Cutting and packing (C&P) problems have gained the attention of geometrical researchers for the past few decades given its applications in a wide variety of industries. Phi-function is a popular tool used in solving these problems. Phi-function for convex polygons is explored in detail in this work. After analysis of the existing equations, it was found that the function yielded same output for different input sets of varying separation among convex polygons. The reason for this anomaly was found and a mitigation strategy is also suggested. Consequently a more accurate phi-function for convex polygons is proposed that addresses the above case and gives better result.

Keywords: Cutting and packing problem, convex polygons, Phi-function, NP-complete

1 Introduction

The cutting and packing (C&P) problem in computational geometry has got wide spread applications in many areas like garment industry, sheet metal cutting, furniture making, etc. All packing problems are characterized by the usage of strong geometrical tools like No Fit Polygon, or trigonometry and generally belong to the class of NP-complete problems\(^1\). Here we are concentrating on the popular tool called as phi-function with special focus on phi function for convex polygons. Convex polygon falls under the category of phi-objects, which are those objects on which phi-functions can be applied.

In this article, the properties of phi-functions are discussed and explained in section 2. The details on the phi-function for convex polygons are described in section 3. In the final section, the anomaly with the existing phi-function for convex polygons is discussed along with the proof, analysis and an equation is proposed for better accuracy.

2 Phi-function fundamentals

Phi functions are one of the most effective tools used in cutting and packing problems, which are often called as nesting problems. This method can be used while we are dealing with phi-objects(as illustrated in\(^3\)). The value of phi-function of a given pair of phi-objects will give the measure of closeness or separation of the objects\(^9\). Its properties state that the value of phi-function will be zero, if both the objects are touching, positive if they are separated and negative when overlapping\(^6\)\(^8\). That is for two phi-objects \(A\) and \(B\), \(\phi^{AB}\) must satisfy

\[
\begin{align*}
\phi^{AB} > 0 & \quad \text{if } A \cap B = \emptyset, \\
\phi^{AB} = 0 & \quad \text{if } \text{int}(A) \cap \text{int}(B) = \emptyset \text{ and fr}(A) \cap \text{fr}(B) \neq \emptyset, \\
\phi^{AB} < 0 & \quad \text{if } \text{int}(A) \cap \text{int}(B) \neq \emptyset.
\end{align*}
\]
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Here fr(x) means the boundary (frontier) and int(x) means the interior of object (x). When $\phi^{AB} > 0$, the magnitude of the phi-function roughly represents the degree of separation between the two phi-objects. Similarly, when $\phi^{AB} < 0$, the magnitude of the phi-function roughly represents the degree of overlap between the two phi-objects [3]. No-Fit Polygon is also a popular tool used in C&P applications. It is observed that No-Fit Polygon (NFP) coincides with the zero level set of phi-function when one object is fixed and rotation is not allowed [3]. A lot of methods have been formulated for creating NFPs; out of which the orbiting algorithm is the first of its kind [5].

3 Phi-function for convex polygons

Phi-functions have been defined for basic shapes like circle, rectangle and convex polygons. For complex shapes like ellipses, a new phi-function called as quasi-phi-function [7] has been defined, where ordinary phi-functions will be too complicated. Complex shapes can be derived from these basic shapes by means of operations like union and intersection [2]. The resultant objects are called composed phi-objects. In this article, phi-function for convex polygons described in [3] is explored in detail and tried to improvise.

Convex polygons are those polygons that satisfy the property that, any two internal points can be connected using a line, which falls completely inside the polygon. A brief explanation of phi-functions described in [3, 4] for convex polygons is given below. Two convex polygons $K'$ and $K''$ can be represented by means of the equations

\[(K', (\alpha'_1, \beta'_1, \gamma'_1), (\alpha'_2, \beta'_2, \gamma'_2), ..., (\alpha'_{m'}, \beta'_{m'}, \gamma'_{m'}))\]  

(2)

and

\[(K'', (\alpha''_1, \beta''_1, \gamma''_1), (\alpha''_2, \beta''_2, \gamma''_2), ..., (\alpha''_{m''}, \beta''_{m''}, \gamma''_{m''}))\]  

(3)

where $m'$ and $m''$ are the number of sides in the polygons $K'$ and $K''$ respectively. Each side of the
polygon is represented by means of three parameters $\alpha, \beta$ and $\gamma$. The general equation for each side is

$$\alpha x + \beta y + \gamma = 0$$  (4)

Each point in $K'$ is represented as $(x'_i, y'_i)$, where $1 \leq i \leq m'$ and similarly those in $K''$ are represented as $(x''_i, y''_i)$, where $1 \leq i \leq m''$. For the ease of computation in the next sections the equation of the line is normalized in such a way that $\alpha^2 + \beta^2 = 1$.

For computing the phi-function, we mainly use the equation for the perpendicular distance between line and a point. Given an edge having parameter $\alpha, \beta$ and $\gamma$ and a point $(x_i, y_i)$ the signed distance is computed as $d = \alpha x_i + \beta y_i + \gamma$ where the sign of $d$ is determined automatically as follows: the sign is negative if point $(x_i, y_i)$ lies on the same side of the edge as the entire polygon and positive otherwise. The magnitude obtained in $d$ is exactly the perpendicular distance between the point and the line as we have already normalized that $\alpha^2 + \beta^2 = 1$. Now let

$$u_{ij} = \alpha'_i x'_j + \beta'_i y'_j + \gamma'_i$$  (5)

denote the signed distance from the $i^{th}$ edge of $K'$ to the $j^{th}$ vertex $(x'_j, y'_j)$ of $K''$ and

$$v_{ji} = \alpha''_j x'_j + \beta''_j y'_j + \gamma''_j$$  (6)

denote the signed distance from the $j^{th}$ edge of $K''$ to the $i^{th}$ vertex $(x'_i, y'_i)$ of $K'$. We can now give the phi-function for the polygons $K'$ and $K''$ as

$$\phi^{K',K''} = \max \left\{ \max_{1 \leq i \leq m'} \min_{1 \leq j \leq m''} u_{ij}, \max_{1 \leq j \leq m''} \min_{1 \leq i \leq m'} v_{ji} \right\}$$  (7)

In the above equation if $K'$ and $K''$ are disjoint then there exist one edge such that these polygons lie on the opposite sides of the line containing that edge. This property in ensures that (7) satisfies the condition mentioned in (1) especially $\phi^{AB} > 0$ if both are disjoint.

4 Special Case

After careful study of the existing phi-function for convex polygons, some anomalies were noted in certain special cases. The authors will explain in detail the special case, the reason for observing anomalies and propose a method for mitigating this issue.

4.1 Proof

Consider the arrangement of two triangles as shown in Fig.1 with vertices defined by A(5,18), B(10,21), C(11,15), D(9,10), E(8,2) and F(11,6). The dashed blue lines are extension to the edges to draw the perpendicular line from an edge to a vertex as it is used in the computation of $u_{ij}$ and $v_{ji}$. To substitute in (7) consider $\Delta DEF$ as $K'$ and $\Delta ABC$ as $K''$. After applying the equations of $u_{ij}$ and $v_{ji}$ we find the $\min_{1 \leq i \leq m'} u_{ij}$ and $\min_{1 \leq j \leq m''} v_{ji}$ in (7) for each edges, which are the distances of the lines $AP$, $CQ$ (represented in orange) and $DR$, $ET$, $ES$ (represented in green) respectively.(Note that edge $DF$ is collinear with $A$ which makes it the closest point to that edge in terms of $u_{ij}$). So we can say that the length of orange lines ($AP$, $CQ$ and $AA$) represent the least signed perpendicular distances from each of the line that forms the edges of $\Delta DEF$ to all the vertices of $\Delta ABC$. Similarly, the length of green lines ($DR$, $ET$ and $ES$) represent the least signed perpendicular distances from each of the line that forms the edges of $\Delta ABC$ to all the vertices of $\Delta DEF$. The sign of each distance is determined as explained in the previous section.
The $\max_{1 \leq i \leq m'} \min_{1 \leq j \leq m''} u_{ij}$ is zero which corresponds to the perpendicular distance from the line that forms the edge $DF$ to the vertex $A$, as the values for $u_{ij}$ associated with edges $DE$ and $FE$ (length of $CQ$ and $AP$ respectively) are negative. Similarly, the $\max_{1 \leq j \leq m''} \min_{1 \leq i \leq m'} v_{ji}$ is the length of the line $DR$, as the values for $v_{ji}$ associated with edges $AB$ and $BC$ (length of $ET$ and $ES$ respectively) are negative. Finally, the phi-function for these two convex polygons will be the maximum of these two values, which in this case is the later one, i.e. the length of line $DR$.

After having found the phi-function for $\Delta ABC$ and $\Delta DEF$ we will now translate $\Delta DEF$ by $(3, -1.5)$ to get a new triangle $\Delta D'E'F'$ as shown in Fig.2. Phi function for $\Delta ABC$ and $\Delta D'E'F'$ is then computed using the same methods.

The $\Delta DEF$ is actually translated along the yellow line which is parallel to the edge $AC$ to obtain $\Delta D'E'F'$. All the notations used in Fig.2 are the same as that in Fig.1. After applying the equations of $u_{ij}$ and $v_{ji}$ we find the $\min_{1 \leq j \leq m''} u_{ij}$ and $\min_{1 \leq i \leq m'} v_{ji}$ in equation (7) for each edges. Here the length of orange lines ($AP$, $CR$ and $AQ$) represent the least signed perpendicular distances from each of the lines that form the edges of $\Delta D'E'F'$ to all the vertices of $\Delta ABC$. Similarly, the length of green lines ($D'S$, $E'U$ and $E'T$) represent the least signed perpendicular distances from each of the lines that form the edges of $\Delta ABC$ to all the vertices of $\Delta D'E'F'$. After performing the calculations it is found that, the orange line having maximal $u_{ij}$ is $CR$ and the green line having maximal $v_{ji}$ is $D'S$. The highest of these two is the phi-function value for the triangles $\Delta ABC$ and $\Delta D'E'F'$, which is obviously $D'S$.

The edge $AC$ is parallel to the line connecting the vertices $D$ and $D'$. Thus, the perpendicular distance from $D$ or $D'$ to the line that forms the edge $AC$ will be the same (property of parallelism), which in turn means that value of phi-function remained the same for different configuration of the same triangles. In Fig.1, the shortest line that can connect both the triangles run between $D$ and $C$. In the case of Fig.2, the shortest line that can connect both the triangles run between $D'$ and $C$. It is quite obvious that $| CD | < | CD' |$, which literally means that $\Delta DEF$ has moved away from $\Delta ABC$. The anomaly observed here is that the magnitude of the phi-function is not giving the degree of separation between these two.
objects. Even though the triangles have moved apart, the phi-function values remained the same.

If the above phi-function for convex polygons are used as a tool in nesting problems, it will treat two different configurations of a pair of polygons equally, whose degree of separation is in turn different.

4.2 Analysis

The anomalies explained in the previous section is analyzed to fully understand the underlying reason for the same. The values of \( u_{ij} \) and \( v_{ji} \) are used for computing phi-function as defined in equation (7). These two values take into consideration, a vertex of first convex polygon and a line that forms the edge of second polygon. It does not take into consideration the end points in the line that defines the edge, instead the whole line is considered. This is the root cause of the anomaly that is observed.

Fig. 3 shows a horizontal line segment \( AB \) and a set of points \( P_1 \) to \( P_5 \). For representing the separation between a point and a line segment, perpendicular distance will be the best choice only in some situations. For example, separation of the points \( P_2 \) and \( P_3 \) from the line segment \( AB \) is the perpendicular distance itself. But \( P_1, P_4 \) and \( P_5 \) lies outside the reach of any perpendicular line drawn from the line segment \( AB \). The method used in equations (5) and (6) will conclude that \( P_4 \) and \( P_5 \) have the same separation from the line segment (edge) \( AB \), as it considers only the equation of the line passing through \( A \) and \( B \) (or the edge) and doesn’t take into account the end points of the edge. This is the reason why there was no change in the phi-function even after the movement of one object away from the other.

As explained above, only for some points, the equations (5) and (6) are valid. So a test condition must be developed to check the validity. In other words, if and only if a perpendicular can be drawn from the edge to the vertex, perpendicular distance is considered, else the distance from the vertex to the nearest end point of the edge must be set as the measure of \( u_{ij} \) or \( v_{ji} \).

Fig. 4 shows a horizontal line segment \( AB \) and a set of points \( P_1 \) to \( P_5 \). For representing the separation between a point and a line segment, perpendicular distance will be the best choice only in some situations. For example, separation of the points \( P_2 \) and \( P_3 \) from the line segment \( AB \) is the perpendicular distance itself. But \( P_1, P_4 \) and \( P_5 \) lies outside the reach of any perpendicular line drawn from the line segment \( AB \). The method used in equations (5) and (6) will conclude that \( P_4 \) and \( P_5 \) have the same separation from the line segment (edge) \( AB \), as it considers only the equation of the line passing through \( A \) and \( B \) (or the edge) and doesn’t take into account the end points of the edge. This is the reason why there was no change in the phi-function even after the movement of one object away from the other.

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So the new phi-function should compute the values of \( u_{ij} \) and \( v_{ji} \) as the distance of dashed lines shown in Fig. 4 for any given vertex \( P_i \) and an edge \( AB \). The rule for determining the sign of \( u_{ij} \) and \( v_{ji} \) remains the same as before. Unlike the previous case here the points \( P_4 \) and \( P_5 \) are having different magnitudes of separation with edge \( AB \).
4.3 Updated Phi-function

Consider Fig. 5 where we want to determine the $u_{ij}$ or $v_{ji}$ values for the edge $AB$ and any point $P_i(x_p, y_p)$. Let $L_A$ and $L_B$ be the perpendiculars to $AB$ drawn through the vertices $A(x_a, y_a)$ and $B(x_b, y_b)$ respectively. Equations (5) and (6) are valid to only those points that lies in between $L_A$ and $L_B$.

Slope of the edge $AB$ is given by

$$ m = \frac{y_a - y_b}{x_a - x_b} \quad (8) $$

Equation of $L_A$:

$$ y - y_a = \frac{-1}{m}(x - x_a) \Rightarrow \alpha_A x + \beta_A y + \gamma_A = 0 \quad (9) $$

Equation of $L_B$:

$$ y - y_b = \frac{-1}{m}(x - x_b) \Rightarrow \alpha_B x + \beta_B y + \gamma_B = 0 \quad (10) $$

For checking whether the point lies in between $L_A$ and $L_B$ the test can be defined as follows: 1) With respect to line $L_A$, point $P$ and point $B$ should be on the same side. 2) With respect to line $L_B$, point $P$ and point $A$ should be on the same side. If both the conditions are satisfied then we can say that the point lies in between $L_A$ and $L_B$. To check this criterion, the conventional equation to find the distance between line and a point is used. Given two points and an equation of a line, then if both the distances are having the same sign we can conclude that both the points lie in the same side of the line. The condition for point $P(x_p, y_p)$ to lie between $L_A$ and $L_B$ can be represented as

$$(\alpha_A x_p + \beta_A y_p + \gamma_A)(\alpha_B x_b + \beta_B y_b + \gamma_B) \geq 0$$

and

$$(\alpha_B x_p + \beta_B y_p + \gamma_B)(\alpha_A x_a + \beta_A y_a + \gamma_A) \geq 0 \quad (11)$$

The product of the distances will be positive if both are having the same sign. Here we observe that the denominator $\sqrt{\alpha^2 + \beta^2}$ is avoided in the distance equation, as sign of the distance is our only concern and not the magnitude. If the conditions in (11) is satisfied then

$$ u_{ij} = \alpha x_p + \beta y_p + \gamma \quad (12) $$

where $\alpha$, $\beta$, and $\gamma$ are the parameters of the line segment $AB$. If the condition (11) is not satisfied then the distance to the closest end point will be considered

$$ u_{ij} = \min \left\{ \frac{\sqrt{(x_a - x_p)^2 + (y_a - y_p)^2}}{\sqrt{(x_b - x_p)^2 + (y_b - y_p)^2}} \right\} \quad (13) $$

The same equations in (12) and (13) are applicable for $v_{ji}$. The sign of both $u_{ij}$ and $v_{ji}$ are determined like before. The final equation for $\phi_{K'K''}$ also remains the same as (7).
5 Conclusion

Phi-functions for convex polygons have applications in cutting and packing problems as they measure the degree of overlap or separation. In this work, the existing phi-functions for convex polygons are studied and analyzed. An optimization is proposed to improve the accuracy of the phi function for convex polygons. With case studies, it is proved that the proposed optimized phi-function gives accurate results compared to the phi-function discussed in[3, 4].

References


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