An Optimization of Phi-function for Convex Polygons

Narendran Rajagopalan[1]*, Mala C.[2], and Anit Nobert[1]
[1]National Institute of Technology Puducherry, Karaikal, India
[2]National Institute of Technology Trichy, Tiruchirapalli, India
narendran@nitpy.ac.in, mala@nitt.edu, anitnobert@gmail.com

Abstract

Cutting and packing (C&P) problems have gained the attention of geometrical researchers for the past few decades given its applications in a wide variety of industries. Phi-function is a popular tool used in solving these problems. Phi-function for convex polygons is explored in detail in this work. After analysis of the existing equations, it was found that the function yielded same output for different input sets of varying separation among convex polygons. The reason for this anomaly was found and a mitigation strategy is also suggested. Consequently a more accurate phi-function for convex polygons is proposed that addresses the above case and gives better result.

Keywords: Cutting and packing problem, convex polygons, Phi-function, NP-complete

1 Introduction

The cutting and packing (C&P) problem in computational geometry has got wide spread applications in many areas like garment industry, sheet metal cutting, furniture making, etc. All packing problems are characterized by the usage of strong geometrical tools like No Fit Polygon, or trigonometry and generally belong to the class of NP-complete problems[1]. Here we are concentrating on the popular tool called as phi-function with special focus on phi function for convex polygons. Convex polygon falls under the category of phi-objects, which are those objects on which phi-functions can be applied.

In this article, the properties of phi-functions are discussed and explained in section 2. The details on the phi-function for convex polygons are described in section 3. In the final section, the anomaly with the existing phi-function for convex polygons is discussed along with the proof, analysis and an equation is proposed for better accuracy.

2 Phi-function fundamentals

Phi functions are one of the most effective tools used in cutting and packing problems, which are often called as nesting problems. This method can be used while we are dealing with phi-objects(as illustrated in[3]). The value of phi-function of a given pair of phi-objects will give the measure of closeness or separation of the objects[9]. Its properties state that the value of phi-function will be zero, if both the objects are touching, positive if they are separated and negative when overlapping[6, 8].That is for two phi-objects A and B, ϕ^{AB} must satisfy

$$\begin{cases} \phi^{AB} > 0 & \text{if } A \cap B = \emptyset, \\ \phi^{AB} = 0 & \text{if } \operatorname{int}(A) \cap \operatorname{int}(B) = \emptyset \text{ and } \operatorname{fr}(A) \cap \operatorname{fr}(B) \neq \emptyset, \\ \phi^{AB} < 0 & \text{if } \operatorname{int}(A) \cap \operatorname{int}(B) \neq \emptyset. \end{cases} \tag{1}$$

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*Corresponding author: Dept. of CSE, National Institute of Technology Puducherry, Karaikal, India-609 609. Tel: +91 95433 82640

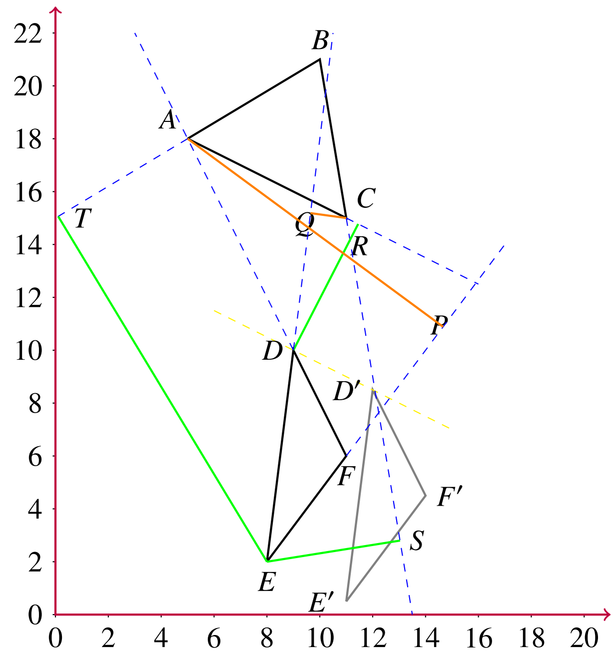

Figure 1: Initial Configuration

Here fr(x) means the boundary (frontier) and int(x) means the interior of object (x). When $\phi^{AB} > 0$, the magnitude of the phi-function roughly represents the degree of separation between the two phi-objects. Similarly, when $\phi^{AB} < 0$, the magnitude of the phi-function roughly represents the degree of overlap between the two phi-objects[3]. No-Fit Polygon is also a popular tool used in C&P applications. It is observed that No-Fit Polygon (NFP) coincides with the zero level set of phi-function when one object is fixed and rotation is not allowed[3]. A lot of methods have been formulated for creating NFPs; out of which the orbiting algorithm is the first of its kind[5].

3 Phi-function for convex polygons

Phi-functions have been defined for basic shapes like circle, rectangle and convex polygons. For complex shapes like ellipses, a new phi-function called as quasi-phi-function[7] has been defined, where ordinary phi-functions will be too complicated. Complex shapes can be derived from these basic shapes by means of operations like union and intersection [2]. The resultant objects are called composed phi-objects. In this article, phi-function for convex polygons described in[3] is explored in detail and tried to improvise.

Convex polygons are those polygons that satisfy the property that, any two internal points can be connected using a line, which falls completely inside the polygon. A brief explanation of phi-functions described in[3, 4] for convex polygons is given below. Two convex polygons K' and K'' can be represented by means of the equations

$$(K', (\alpha'_1, \beta'_1, \gamma'_1), (\alpha'_2, \beta'_2, \gamma'_2), ..., (\alpha'_{m'}, \beta'_{m'}, \gamma'_{m'})) \tag{2}$$

and

$$(K'', (\alpha''_1, \beta''_1, \gamma''_1), (\alpha''_2, \beta''_2, \gamma''_2), ..., (\alpha''_{m''}, \beta''_{m''}, \gamma''_{m''})) \tag{3}$$

where m' and m'' are the number of sides in the polygons K' and K'' respectively. Each side of the polygon is represented by means of three parameters α,β and γ. The general equation for each side is

$$\alpha x + \beta y + \gamma = 0 \tag{4}$$

Each point in K' is represented as (x'_i, y'_i), where $1 \le i \le m'$ and similarly those in K'' are represented as (x''_i, y''_i), where $1 \le i \le m''$. For the ease of computation in the next sections the equation of the line is normalized in such a way that $\alpha^2 + \beta^2 = 1$.

For computing the phi-function, we mainly use the equation for the perpendicular distance between line and a point. Given an edge having parameter α,β and γ and a point (x_i, y_i) the signed distance is computed as $d = \alpha x_i + \beta y_i + \gamma$ where the sign of d is determined automatically as follows: the sign is negative if point (x_i, y_i) lies on the same side of the edge as the entire polygon and positive otherwise. The magnitude obtained in d is exactly the perpendicular distance between the point and the line as we have already normalized that $\alpha^2 + \beta^2 = 1$. Now let

$$u_{ij} = \alpha'_i x''_j + \beta'_i y''_j + \gamma'_i \tag{5}$$

denote the signed distance from the i^{th} edge of K' to the j^{th} vertex (x''_j, y''_j) of K'' and

$$v_{ji} = \alpha''_j x'_i + \beta''_j y'_i + \gamma''_j \tag{6}$$

denote the signed distance from the j^{th} edge of K'' to the i^{th} vertex (x'_i, y'_i) of K'. We can now give the phi-function for the polygons K' and K'' as

$$\phi^{K'K''} = \max\left\{ \max_{1\le i\le m'} \min_{1\le j\le m''} u_{ij}, \max_{1\le j\le m''} \min_{1\le i\le m'} v_{ji} \right\} \tag{7}$$

In the above equation if K' and K'' are disjoint then there exist one edge such that these polygons lie on the opposite sides of the line containing that edge. This property in ensures that (7) satisfies the condition mentioned in (1) especially $\phi^{AB} > 0$ if both are disjoint.

4 Special Case

After careful study of the existing phi-function for convex polygons, some anomalies were noted in certain special cases. The authors will explain in detail the special case, the reason for observing anomalies and propose a method for mitigating this issue.

4.1 Proof

Consider the arrangement of two triangles as shown in Fig.1 with vertices defined by $A(5,18)$, $B(10,21)$, $C(11,15)$, $D(9,10)$, $E(8,2)$ and $F(11,6)$. The dashed blue lines are extension to the edges to draw the perpendicular line from an edge to a vertex as it is used in the computation of u_{ij} and v_{ji}. To substitute in (7) consider ΔDEF as K' and ΔABC as K''. After applying the equations of u_{ij} and v_{ji} we find the $\min_{1\le j\le m''} u_{ij}$ and $\min_{1\le i\le m'} v_{ji}$ in (7) for each edges, which are the distances of the lines AP, CQ (represented in orange) and DR, ET, ES (represented in green) respectively.(Note that edge DF is collinear with A which makes it the closest point to that edge in terms of u_{ij}). So we can say that the length of orange lines (AP, CQ and AA) represent the least *signed* perpendicular distances from each of the line that forms the edges of ΔDEF to all the vertices of ΔABC. Similarly, the length of green lines (DR, ET and ES) represent the least *signed* perpendicular distances from each of the line that forms the edges of ΔABC to all the vertices of ΔDEF. The sign of each distance is determined as explained in the previous section.

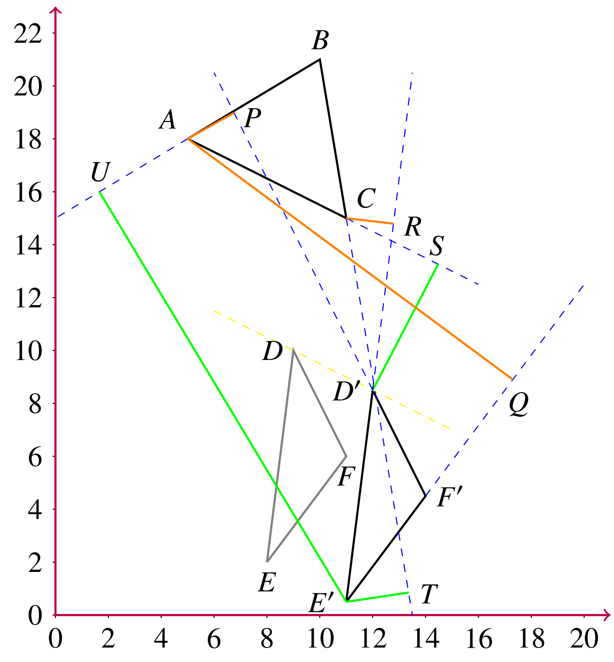

Figure 2: Final Configuration

The $\max\limits_{1\le i\le m'}\min\limits_{1\le j\le m''} u_{ij}$ is zero which corresponds to the perpendicular distance from the line that forms the edge DF to the vertex A, as the values for u_{ij} associated with edges DE and FE (length of CQ and AP respectively)are negative. Similarly, the $\max\limits_{1\le j\le m''}\min\limits_{1\le i\le m'} v_{ji}$ is the length of the line DR, as the values for v_{ji} associated with edges AB and BC (length of ET and ES respectively) are negative. Finally the phi-function for these two convex polygons will be the maximum of these two values, which in this case is the later one, i.e. the length of line DR.

After having found the phi-function for ΔABC and ΔDEF we will now translate ΔDEF by $(3,-1.5)$ to get a new triangle $\Delta D'E'F'$ as shown in Fig.2. Phi function for ΔABC and $\Delta D'E'F'$ is then computed using the same methods.

The ΔDEF is actually translated along the yellow line which is parallel to the edge AC to obtain $\Delta D'E'F'$. All the notations used in Fig.2 are the same as that in Fig.1. After applying the equations of u_{ij} and v_{ji} we find the $\min\limits_{1\le j\le m''} u_{ij}$ and $\min\limits_{1\le i\le m'} v_{ji}$ in equation (7) for each edges. Here the length of orange lines (AP, CR and AQ) represent the least *signed* perpendicular distances from each of the lines that form the edges of $\Delta D'E'F'$ to all the vertices of ΔABC. Similarly, the length of green lines ($D'S$, $E'U$ and $E'T$) represent the least *signed* perpendicular distances from each of the lines that form the edges of ΔABC to all the vertices of $\Delta D'E'F'$. After performing the calculations it is found that, the orange line having maximal u_{ij} is CR and the green line having maximal v_{ji} is $D'S$. The highest of these two is the phi-function value for the triangles ΔABC and $\Delta D'E'F'$, which is obviously $D'S$.

The edge AC is parallel to the line connecting the vertices D and D'. Thus, the perpendicular distance from D or D' to the line that forms the edge AC will be the same (property of parallelism), which in turn means that value of phi-function remained the same for different configuration of the same triangles. In Fig.1, the shortest line that can connect both the triangles run between D and C. In the case of Fig.2, the shortest line that can connect both the triangles run between D' and C. It is quite obvious that $|CD|<|CD'|$, which literally means that ΔDEF has moved away from ΔABC. The anomaly observed here is that the magnitude of the phi-function is not giving the degree of separation between these two objects. Even though the triangles have moved apart, the phi-function values remained the same.

If the above phi-function for convex polygons are used as a tool in nesting problems, it will treat two different configurations of a pair of polygons equally, whose degree of separation is in turn different.

4.2 Analysis

The anomalies explained in the previous section is analyzed to fully understand the underlying reason for the same. The values of u_{ij} and v_{ji} are used for computing phi-function as defined in equation (7). These two values take into consideration, a vertex of first convex polygon and a *line that forms the edge* of second polygon. It does not take into consideration the end points in the line that defines the edge, instead the whole line is considered. This is the root cause of the anomaly that is observed.

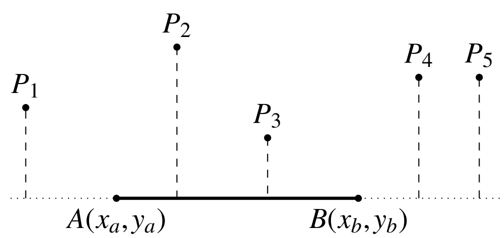

Figure 3: Distance calculation without considering edge end points

Fig.3 shows a horizontal line segment AB and a set of points P_1 to P_5. For representing the separation between a point and a line segment, perpendicular distance will be the best choice only in some situations. For example, separation of the points P_2 and P_3 from the line segment AB is the perpendicular distance itself. But P_1, P_4 and P_5 lies outside the reach of any perpendicular line drawn from the line segment AB. The method used in equations (5) and (6) will conclude that P_4 and P_5 have the same separation from the line segment (edge) AB, as it considers only the equation of the line passing through A and B (or the edge) and doesn't take into account the end points of the edge. This is the reason why there was no change in the phi-function even after the movement of one object away from the other.

As explained above, only for some points, the equations (5) and (6) are valid. So a test condition must be developed to check the validity. In other words, if and only if a perpendicular can be drawn from the edge to the vertex, perpendicular distance is considered, else the distance from the vertex to the nearest end point of the edge must be set as the measure of u_{ij} or v_{ji}.

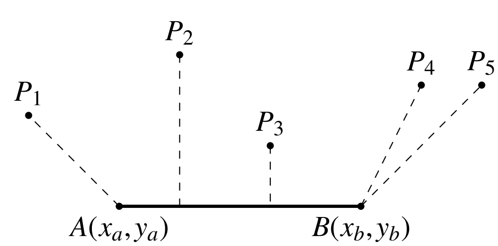

Figure 4: Distance calculation after considering edge end points

So the new phi-function should compute the values of u_{ij} and v_{ji} as the distance of dashed lines shown in Fig.4, for any given vertex P_i and an edge AB. The rule for determining the sign of u_{ij} and v_{ji} remains the same as before. Unlike the previous case here the points P_4 and P_5 are having different magnitudes of separation with edge AB.

4.3 Updated Phi-function

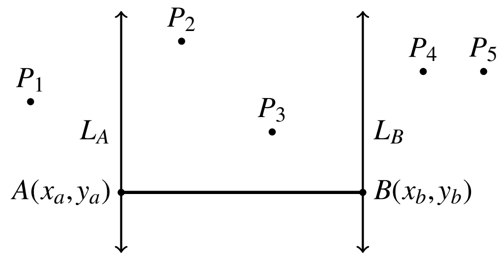

Figure 5: Testing condition for updated phi-function

Consider Fig.5, where we want to determine the u_{ij} or v_{ji} values for the edge AB and any point $P_i(x_p, y_p)$. Let L_A and L_B be the perpendiculars to AB drawn through the vertices $A(x_a, y_a)$ and $B(x_b, y_b)$ respectively. Equations (5) and (6) are valid to only those points that lies in between L_A and L_B.
Slope of the edge AB is given by

$$m = \frac{y_a - y_b}{x_a - x_b} \tag{8}$$

Equation of L_A:

$$y - y_a = \frac{-1}{m}(x - x_a) \Rightarrow \alpha_A x + \beta_A y + \gamma_A = 0 \tag{9}$$

Equation of L_B:

$$y - y_b = \frac{-1}{m}(x - x_b) \Rightarrow \alpha_B x + \beta_B y + \gamma_B = 0 \tag{10}$$

For checking whether the point lies in between L_A and L_B the test can be defined as follows: 1) With respect to line L_A, point P and point B should be on the same side. 2) With respect to line L_B, point P and point A should be on the same side. If both the conditions are satisfied then we can say that the point lies in between L_A and L_B. To check this criterion, the conventional equation to find the distance between line and a point is used. Given two points and an equation of a line, then if both the distances are having the same sign we can conclude that both the points lie in the same side of the line. The condition for point $P(x_p, y_p)$ to lie between L_A and L_B can be represented as

$$\begin{aligned} (\alpha_A x_p + \beta_A y_p + \gamma_A)(\alpha_A x_b + \beta_A y_b + \gamma_A) \geq 0 \\ \text{and} \\ (\alpha_B x_p + \beta_B y_p + \gamma_B)(\alpha_B x_a + \beta_B y_a + \gamma_B) \geq 0 \end{aligned} \tag{11}$$

The product of the distances will be positive if both are having the same sign. Here we observe that the denominator $\sqrt{\alpha^2 + \beta^2}$ is avoided in the distance equation, as sign of the distance is our only concern and not the magnitude. If the conditions in (11) is satisfied then

$$u_{ij} = \alpha x_p + \beta y_p + \gamma \tag{12}$$

where α β and γ are the parameters of the line segment AB. If the condition (11) is not satisfied then the distance to the closest end point will be considered

$$u_{ij} = \min \begin{cases} \sqrt{(x_a - x_p)^2 + (y_a - y_p)^2} \\ \sqrt{(x_b - x_p)^2 + (y_b - y_p)^2} \end{cases} \tag{13}$$

The same equations in (12) and (13) are applicable for v_{ji}. The sign of both u_{ij} and v_{ji} are determined like before. The final equation for $\phi^{K'K''}$ also remains the same as (7).

5 Conclusion

Phi-functions for convex polygons have applications in cutting and packing problems as they measure the degree of over-lap or separation. In this work, the existing phi-functions for convex polygons are studied and analyzed. An optimization is proposed to improve the accuracy of the phi function for convex polygons. With case studies, it is proved that the proposed optimized phi-function gives accurate results compared to the phi-function discussed in[3, 4].

References

[1] J. A. Bennell and J. F. Oliveira. The geometry of nesting problems: A tutorial. *European Journal of Operational Research*, 184(2):397–415, January 2008.

[2] J. A. Bennell and J. F. Oliveira. A tutorial in irregular shape packing problems. *Journal of the Operational Research Society*, 60(Supplement 1):S93–S105, February 2009.

[3] N. Chernov, Y. Stoyan, and T. Romanova. Mathematical model and efficient algorithms for object packing problem. *Computational Geometry*, 43(5):535–553, July 2010.

[4] N. Chernov, Y. Stoyan, T. Romanova, and A. Pankratov. Phi-functions for 2d objects formed by line segments and circular arcs. *Advances in Operations Research*, 2012:346358:1–346358:26, January 2012.

[5] A. Mahadevan. Optimization in computer-aided pattern packing (marking, envelopes), 1984.

[6] G. Scheithauer, Y. G. Stoyan, and T. Y. Romanova. Mathematical modeling of interactions of primary geometric 3d objects. *Cybernetics and Systems Analysis*, 41(3):332–342, May 2005.

[7] Y. Stoyan, A. Pankratov, and T. Romanova. Quasi-phi-functions and optimal packing of ellipses. *Journal of Global Optimization*, 65(2):283–307, June 2016.

[8] Y. Stoyan, G. Scheithauer, N. Gil, and T. Romanova. Phi-functions for complex 2d-objects. *Quarterly Journal of the Belgian, French and Italian Operations Research Societies*, 2(1):69–84, 2004.

[9] Y. G. Stoyan. Mathematical methods for geometric design. In *Proc. of the IFIP 9th World Computer Congress, Paris, France*, volume 82, pages 67–86, September 1983.

Author Biography

Narendran Rajagopalan received the Master's degree from Sri Jayachamarajendra College of Engineering, Mysore, India in 2007. He completed Ph.D degree in the Department of Computer Science and Engineering at National Institute of Technology, Trichy, India in 2013 and is currently working as assistant professor in the department of computer science and engineering, National Institute of Technology, Puducherry. His research interests include Wireless Networking, Security, Quality of Service and Soft Computing.

Mala C. is a Professor in the Department of Computer Science and Engineering, National Institute of Technology, Tiruchirappalli, Tamil Nadu, India – 620 015. Her research area of interest includes Data Structures & Algorithms, Computer Networks, Parallel Algorithms, Computer Architecture, Sensor Networks, Soft Computing Techniques, Image Processing, Intelligent Transportation Systems and Vehicular Adhoc Networks.

Anit Nobert is a Computer Science and Engineering post graduate from National Institute of Technology Puducherry graduated in the year 2007. He is currently employed as a research engineer at Trimble India Inc., and his research interests include computational geometry and parallel computing with Nvidia GPUs.